On the preconditioned AOR iterative method for Z-matrices

Davod Khojasteh Salkuyeh[†1] and Mohsen Hasani[‡]

[†]*Department of Mathematics, University of Mohaghegh Ardabili,*
P.O.Box. 179, Ardabil, Iran
email: khojaste@uma.ac.ir
[‡]*Faculty of Science, Department of Mathematics, Islamic Azad University,*
Shahrood, Iran
email: hasani.mo@gmail.com

Abstract

Several preconditioned AOR methods have been proposed to solve system of linear equations $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a unit Z-matrix. The aim of this paper is to give a comparison result for a class of preconditioners P, where $P \in \mathbb{R}^{n\times n}$ is nonsingular, nonnegative and has unit diagonal entries. Numerical results for corresponding preconditioned GMRES methods are given to illustrate the theoretical results.

Key words: System of linear equations, Preconditioner, AOR iterative method, Z-matrix, Comparison.
2000 MSC: 65F10.

1. Introduction

Consider the system of linear equations

$$Ax = b, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. A general stationary iterative method for solving Eq. (1) may be expressed as

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \ldots,$$

in which $A = M - N$, where $M, N \in \mathbb{R}^{n\times n}$ and M is nonsingular. It is well-known that this iterative method is convergent if and only if $\rho(M^{-1}N) < 1$, where $\rho(X)$ refers to the spectral radius of matrix X. Let $a_{ii} \neq 0,\ i = 1, 2, \ldots, n$. Therefore,

[1]Corresponding author

without loss of generality we can assume that $a_{ii} = 1,\ i = 1, 2, \ldots, n$. In this case, we split A into

$$A = I - L - U, \tag{2}$$

where I is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular matrices, respectively. The accelerated overrelaxation (AOR) iterative method to solve Eq. (1) is defined by [4, 12]

$$x^{(k+1)} = \mathcal{L}_{\gamma,\omega} x^{(k)} + \omega(I - \gamma L)^{-1} b,$$

in which

$$\mathcal{L}_{\gamma,\omega} = (I - \gamma L)^{-1}[(1 - \omega)I + (\omega - \gamma)L + \omega U],$$

where ω and γ are real parameters and with $\omega \neq 0$. For certain values of the parameters ω and γ the AOR iterative method results in the Jacobi, Gauss-Seidel and the SOR methods [4].

To improve the convergence rate of an iterative method one may apply it to the preconditioned linear system $PAx = Pb$, where the matrix P is called a preconditioner. Several preconditioners have been presented for the stationary iterative methods by many authors. Recently, Wang and Song in [13] have proposed a general preconditioner P which is nonsingular, nonnegative and has unit diagonal entries. They also have investigated the properties of the preconditioners of the form

$$P = (p_{ij}) = (-\alpha_{ij} a_{ij}), \tag{3}$$

where $0 \leq \alpha_{ij} \leq 1$, for $i \neq j$, and $p_{ii} = 1$ for $i = 1, \ldots, n$. Many preconditioners proposed in the literature are the special cases of such general preconditioner (see for example [3, 5, 6, 7, 8, 9, 13, 15]). In this paper, we show that under some conditions, preconditioner (3) with $\alpha_{ij} = 1$ for $1 \leq i \neq j \leq n$ is the best one among the preconditioners of the form (3).

For convenience, we first present some notations, definitions and preliminaries which will be used in this paper. A matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is said to be nonnegative and denoted by $A \geq 0$ if $a_{ij} \geq 0$ for all i and j and A is said to positive and denoted by $A \gg 0$ if $a_{ij} > 0$ for all i and j. If $A \geq 0$, then by the Perron-Frobenius theory (see for example [2]), $\rho(A)$ is an eigenvalue of A, and corresponding to $\rho(A)$, A has a nonnegative eigenvector, which we refer to as a Perron vector of A.

Definition 1.1. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is an *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$.

Definition 1.2. A Z-matrix A is said to be an *M-matrix* if A is nonsingular and $A^{-1} \geq 0$.

Definition 1.3. Let $A \in \mathbb{R}^{n\times n}$. The representation $A = M - N$ is called a *splitting* of A if M is nonsingular. The splitting $A = M - N$ is called
(a) *convergent* if $\rho(M^{-1}N) < 1$;
(b) *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;
(c) an *M-splitting* of A if M is an M-matrix and $N \geq 0$.

Definition 1.4. A real matrix A is called *monotone* if $Ax \geq 0$ implies $x \geq 0$.

Lemma 1.1. [9, Lemma 3.2] *Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

Lemma 1.2. [13, Lemma 2.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix. Then, there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, $A(\epsilon) = (a_{ij}(\epsilon))$ is also an M-matrix, where*

$$a_{ij}(\epsilon) = \begin{cases} a_{ij}, & if\ a_{ij} \neq 0, \\ -\epsilon, & if\ a_{ij} = 0. \end{cases}$$

Lemma 1.3. [1, Lemma 6.1] *A is monotone if and only if A is nonsingular with $A^{-1} \geq 0$.*

Lemma 1.4. [14, Lemma 1.6] *Let A be a Z-matrix. Then, A is an M-matrix if and only if there is a positive vector x such that $Ax \gg 0$.*

Lemma 1.5. [10, Lemma 2.2] *Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splittings of the monotone matrices A_1 and A_2, respectively, such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector x such that $0 \leq A_1x \leq A_2x$, then for the monotonic norm associated with x,*

$$\|M_2^{-1}N_2\|_x \leq \|M_1^{-1}N_1\|_x.$$

In particular, if $M_1^{-1}N_1$ has a positive Perron vector, then

$$\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_2).$$

2. Main results

Let $A \in \mathbb{R}^{n\times n}$. We consider the preconditioner $\widetilde{P} = (\tilde{p}_{ij})$ for Eq. (1), where

$$\tilde{p}_{ij} = \begin{cases} -\alpha_{ij}a_{ij}, & \text{if } i \neq j, \\ 1, & \text{otherwise}, \end{cases}$$

and $\alpha_{ij} \in \mathbb{R}$ for $i \neq j$. We split $\tilde{P}$ into $\tilde{P} = I + L(\alpha) + U(\alpha)$, where I is the identity matrix and $L(\alpha)$ and $U(\alpha)$ are strictly lower and strictly upper triangular matrices, respectively. Let $\tilde{A} = \tilde{P}A = (I + L(\alpha) + U(\alpha))A$ and

$$\begin{aligned} L(\alpha)U &= G_1(\alpha) + E_1(\alpha) + F_1(\alpha), \\ U(\alpha)L &= G_2(\alpha) + E_2(\alpha) + F_2(\alpha), \end{aligned}$$

where $G_1(\alpha)$ and $G_2(\alpha)$ are diagonal matrices, $F_1(\alpha)$ and $F_2(\alpha)$ are strictly lower triangular matrices and $E_1(\alpha)$ and $E_2(\alpha)$ are strictly upper triangular matrices. In this case, $\tilde{A}$ can be split as $\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$, where $\tilde{D}$, $\tilde{L}$ and $\tilde{U}$, respectively, are diagonal, strictly lower and strictly upper triangular matrices defined as

$$\begin{aligned} \tilde{D} &= I - E_1(\alpha) - E_2(\alpha), \\ \tilde{L} &= L - L(\alpha) + L(\alpha)L + F_1(\alpha) + F_2(\alpha), \\ \tilde{U} &= U + G_1(\alpha) - U(\alpha) + G_2(\alpha) + U(\alpha)U. \end{aligned}$$

If the matrix $\tilde{D} - r\tilde{L}$ is nonsingular, the AOR iteration matrix to solve the preconditioned system $\tilde{P}Ax = \tilde{P}b$ can be written as

$$\tilde{\mathcal{L}}_{\gamma,\omega} = (\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega)\tilde{D} + (\omega - \gamma)\tilde{L} + \omega\tilde{U}].$$

Theorem 2.1. *Let A be a Z-matrix and $\alpha_{ij} \in [0,1]$ for $1 \leq i \neq j \leq n$. Then, A is an M-matrix if and only if $\tilde{A}$ is and M-matrix.*

Proof. The proof of this theorem is similar to the proof of Lemma 3.3 in [9]. Let A be an M-matrix and $\tilde{A} = (\tilde{a}_{ij})$. Then

$$\tilde{a}_{ij} = \begin{cases} 1 - \displaystyle\sum_{k=1, k\neq j}^{n} \alpha_{ik} a_{ik} a_{ki}, & 1 \leq i = j \leq n, \\ a_{ij} - \displaystyle\sum_{k=1, k\neq j}^{n} \alpha_{ik} a_{ik} a_{kj}, & 1 \leq i \neq j \leq n. \end{cases} \tag{4}$$

Since A is a Z-matrix, we have $\tilde{a}_{ij} \leq 0$, for $i \neq j$. This means that $\tilde{A}$ is also a Z-matrix. By Lemma 1.4 there exists a positive vector x such that $Ax \gg 0$. On the other hand, $\tilde{A} = (I + L(\alpha) + U(\alpha))Ax \gg 0$. Invoking Lemma 1.4 implies that $\tilde{A}$ is also an M-matrix.

Conversely, let $\tilde{A}$ be an M-matrix. Then, $\tilde{A}^T$ is also an M-matrix. By Lemma 1.4 there exists a positive vector x such that $\tilde{A}^T x \gg 0$, i.e., $A^T(I + L(\alpha)^T + U(\alpha)^T)x \gg 0$. Let $y = (I + L(\alpha)^T + U(\alpha)^T)x$. Obviously $y \gg 0$. Therefore by Lemma 1.4, A^T is an M-matrix. As a result, A is an M-matrix, as well. $\square$

Theorem 2.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular Z-matrix, $0 \le \gamma \le \omega \le 1$, $\omega \ne 0$ and $\alpha_{ij} \in [0,1]$ for $1 \le i \ne j \le n$. If $\rho(L_{\gamma,\omega}) < 1$, then $\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \le \rho(\mathcal{L}_{\gamma,\omega}) < 1$.*

Proof. Under the assumptions of the theorem, it is easy to see that

$$A = M - N = \frac{1}{\omega}(I - \gamma L) - \frac{1}{\omega}[(1-\omega)I + (\omega-\gamma)L + \omega U),$$

is an M-splitting of A. On the other hand, we have $\rho(M^{-1}N) = \rho(\mathcal{L}_{\gamma,\omega}) < 1$. Therefore, from Lemma 1.1 we deduce that A is an M-matrix. Now, the result follows immediately from Theorems 2.6 and 2.7 in [13]. $\square$

In the sequel, we show that among the preconditioners of the form $\tilde{P} = I + L(\alpha) + U(\alpha)$ with $\alpha_{ij} \in [0,1]$, the preconditioner $\hat{P} = I + L + U$ is the best one to speed up the convergence rate of the AOR iterative method. We mention that, if we assume $\alpha_{ij} = 1$, for $1 \le i \ne j \le n$, then the preconditioner $\tilde{P}$ results in the preconditioner $\hat{P}$. Let the AOR iteration matrix of the preconditioned system $\hat{P}Ax = \hat{P}b$ be

$$\hat{\mathcal{L}}_{\gamma,\omega} = (\hat{D} - \gamma\hat{L})^{-1}[(1-\omega)\hat{D} + (\omega-\gamma)\hat{L} + \omega\hat{U}],$$

where $\hat{A} = \hat{P}A = \hat{D} - \hat{L} - \hat{U}$ in which $\hat{D}$, $\hat{L}$ and $\hat{U}$ are the diagonal, strictly lower and strictly upper triangular matrices, respectively.

Theorem 2.3. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular Z-matrix. Let also $0 \le \gamma \le \omega \le 1$, $\omega \ne 0$ and $\alpha_{ij} \in [0,1]$ for $1 \le i \ne j \le n$. If $\rho(\mathcal{L}_{\gamma,\omega}) < 1$ and*

$$(\alpha_{ij}-1)a_{ij} + (\sum_{k=1,k\ne i}^{n} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1,k\ne i}^{n} a_{ik}a_{kj}) + (\sum_{k=1}^{i-1} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1}^{i-1} a_{ik}a_{kj}) \le 0,$$

for $1 \le j < i \le n$, then $\rho(\hat{\mathcal{L}}_{\gamma,\omega}) \le \rho(\tilde{\mathcal{L}}_{\gamma,\omega})$.

Proof. Let

$$\begin{aligned}
M &= \frac{1}{\omega}(I - \gamma L),\\
N &= \frac{1}{\omega}[(1-\omega)I + (\omega-\gamma)L + \omega U],\\
\tilde{M} &= \frac{1}{\omega}(\tilde{D} - \gamma\tilde{L}),\\
\tilde{N} &= \frac{1}{\omega}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}],
\end{aligned}$$

$$\hat{M} = \frac{1}{\omega}(\hat{D} - \gamma\hat{L}),$$
$$\hat{N} = \frac{1}{\omega}[(1-\omega)\hat{D} + (\omega-\gamma)\hat{L} + \omega\hat{U}].$$

It is easy to see that

$$A = M - N, \qquad \tilde{A} = \tilde{M} - \tilde{N}, \qquad \hat{A} = \hat{M} - \hat{N}.$$

Similar to the proof of Theorem 2.2 we see that the matrix A is an M-matrix. Let $x = A^{-1}e$, where $e = (1,\ldots,1)^T$. We have $x \gg 0$, since none of the rows of A^{-1} can be zero. Therefore,

$$(\hat{A} - \tilde{A})x = [(L - L(\alpha)) + (U - U(\alpha))]Ax = [(L - L(\alpha)) + (U - U(\alpha))]e \geq 0.$$

Since $\rho(\mathcal{L}_{\gamma,\omega}) < 1$, from Theorem 2.2 we have $\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) \leq \rho(\mathcal{L}_{\gamma,\omega}) < 1$. Now, Theorem 2.1 shows that $\tilde{A}$ is an M-matrix. Hence, $\tilde{D}$ is a diagonal matrix with positive diagonal entries. Therefore, $\tilde{M}$ and $\hat{M}$ are nonsingular matrices and splittings $\hat{A} = \hat{M} - \hat{N}$ and $\tilde{A} = \tilde{M} - \tilde{N}$ are M-splitting.

By definition of $\tilde{A}$ and $\hat{A}$, we have

$$\tilde{D} - \widehat{D} = E_1(1) - E_1(\alpha) + E_2(1) - E_2(\alpha) \geq 0.$$

So $\tilde{D} \geq \hat{D}$. On the other hand, we have

$$\tilde{L} - \widehat{L} = (L - L(\alpha)) + (L(\alpha) - L)L + (F_2(\alpha) - F_2(1)) + (F_1(\alpha) - F_1(1)).$$

Hence,

$$\begin{aligned}
(\tilde{L} - \widehat{L})_{ij} &= (\alpha_{ij} - 1)a_{ij} + (\sum_{k=1}^{i-1} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1}^{i-1} a_{ik}a_{kj}) \\
&\quad + (\sum_{k=i+1}^{n} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=i+1}^{n} a_{ik}a_{kj}) + (\sum_{k=1}^{i-1} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1}^{i-1} a_{ik}a_{kj}) \\
&= (\alpha_{ij} - 1)a_{ij} + (\sum_{k=1,k\neq i}^{n} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1,k\neq i}^{n} a_{ik}a_{kj}) \\
&\quad + (\sum_{k=1}^{i-1} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1}^{i-1} a_{ik}a_{kj}) \leq 0,
\end{aligned}$$

which shows that $\tilde{L} \leq \hat{L}$. This result together with $\tilde{D} \geq \hat{D}$ gives $\hat{D} - \gamma\hat{L} \leq \tilde{D} - \gamma\tilde{L}$. Since $\gamma\tilde{D}^{-1}\tilde{L} \geq 0$ is an strictly lower triangular matrix and $\rho(\gamma\tilde{D}^{-1}\tilde{L}) < 1$, we deduce that

$$(\tilde{D} - \gamma\tilde{L})^{-1} = (I - \gamma\tilde{D}^{-1}\tilde{L})^{-1}\tilde{D}^{-1} = I + \sum_{j=1}^{\infty} (\gamma\tilde{D}^{-1}\tilde{L})^j \tilde{D}^{-1} \geq 0.$$

In the same way, $(\hat{D}-\gamma\hat{L})^{-1} \geq 0$. Therefore, we obtain

$$(\tilde{D}-\gamma\tilde{L})^{-1} \leq (\widehat{D}-\gamma\widehat{L})^{-1},$$

and this means that

$$0 \leq \tilde{M}^{-1} \leq \hat{M}^{-1}.$$

Let A be an irreducible matrix. Having in mind that the entries of $\tilde{A}$ are given by (4), we conclude that $\tilde{A}$ is also an irreducible matrix. We have

$$\begin{aligned}
\tilde{\mathcal{L}}_{\gamma,\omega} &= (\tilde{D}-\gamma\tilde{L})^{-1}[(1-\omega)\tilde{D}+(w-\gamma)\tilde{L}+\omega\tilde{U}] \\
&= (I-\gamma\tilde{D}^{-1}\tilde{L})^{-1}[(1-\omega)I+(\omega-\gamma)\tilde{D}^{-1}\tilde{L}+\omega\tilde{D}^{-1}\tilde{U}] \\
&= [\ I+(\gamma\tilde{D}^{-1}\tilde{L})+(\gamma\tilde{D}^{-1}\tilde{L})^2+\cdots\ ][(1-\omega)I+(\omega-\gamma)\tilde{D}^{-1}\tilde{L}+\omega\tilde{D}^{-1}\tilde{U}] \\
&\geq [(1-\omega)I+(\omega-\gamma)\tilde{D}^{-1}\tilde{L}+\omega\tilde{D}^{-1}\tilde{U}].
\end{aligned}$$

This shows that for every $0 \leq \gamma < 1$ the matrix $\tilde{\mathcal{L}}_{\gamma,\omega}$ is a nonnegative irreducible matrix. Hence, from Theorem 4.11 in [1], $\tilde{\mathcal{L}}_{\gamma,\omega} = \tilde{M}^{-1}\tilde{N}$ has a positive Perron vector and from Lemma 1.5 we have

$$\rho(\hat{\mathcal{L}}_{\gamma,\omega}) \leq \rho(\tilde{\mathcal{L}}_{\gamma,\omega}).$$

If $\gamma = 1$, then $\omega = \gamma = 1$ and we have

$$\rho(\tilde{\mathcal{L}}_{1,1}) = \lim_{\gamma\to 1^-} \rho(\tilde{\mathcal{L}}_{\gamma,1}) \geq \lim_{\gamma\to 1^-} \rho(\hat{\mathcal{L}}_{\gamma,1}) = \rho(\hat{\mathcal{L}}_{1,1}).$$

Now, if A is a reducible matrix, then by Lemma 1.2, for sufficiently $\epsilon > 0$ the matrix $A(\epsilon)$ is an irreducible M-matrix and one can see that

$$\rho(\tilde{\mathcal{L}}_{\gamma,\omega}) = \lim_{\epsilon\to 0^+} \rho(\tilde{\mathcal{L}}_{\gamma,\omega}(\epsilon)) \geq \lim_{\epsilon\to 0^+} \rho(\widehat{\mathcal{L}}_{\gamma,\omega}(\epsilon)) = \rho(\widehat{\mathcal{L}}_{\gamma,\omega}). \qquad \square$$

3. Numerical experiments

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a Pentium 4 PC, with a 3.06 GHz CPU and 1.00GB of RAM.

Example 1. We consider the two dimensional convection-diffusion equation (see [14])

$$-(u_{xx}+u_{yy})+u_x+2u_y = f(x,y), \quad \text{in} \quad \Omega=(0,1)\times(0,1), \tag{5}$$

with the homogeneous Dirichlet boundary conditions. Discretization of this equation on a uniform grid with $N \times N$ interior nodes ($n = N^2$), by using the second

order centered differences for the second and first order differentials gives a linear system of equations of order n with n unknowns. The coefficient matrix of the obtained system is of the form

$$A = I \otimes P + Q \otimes I,$$

where $\otimes$ denotes the Kronecker product,

$$P = \text{tridiag}(-\frac{2+h}{8}, 1, -\frac{2-h}{8}) \quad \text{and} \quad Q = \text{tridiag}(-\frac{1+h}{4}, 0, -\frac{1-h}{4}),$$

are $N \times N$ tridiagonal matrices, and the step size is $h = 1/N$. We consider four preconditioners of the form

$$\begin{aligned}
P_0 &= I, \\
P_1 &= I + 0.5L, \\
P_2 &= I + 0.5L + 0.5U, \\
P_3 &= I + L(\alpha) + U(\alpha), \\
P_4 &= I + L + U,
\end{aligned}$$

where for the preconditioner P_3, α_{ij}'s are random numbers uniformly distributed in the interval $(0, 1)$. We mention that $P_0 = I$ means that no preconditioner is used. In Table 1, the spectral radius of the AOR iterative method applied to the preconditioned systems $P_iAx = P_ib$, $i = 0, \ldots, 4$ for different values of γ, ω and n are given. As we observe preconditioner P_4 is the best one among the chosen preconditioners.

For more investigation, we apply the GMRES(m) method [11] with $m = 10$ to solve $P_iAx = P_ib$, $i = 0, \ldots, 4$. In all the experiments, vector $b = A(1, 1, \ldots, 1)^T$ was taken to be the right-hand side of the linear system and a null vector as an initial guess. The stopping criterion used was always

$$\frac{\|b - Ax_k\|_2}{\|b\|_2} < 10^{-10}.$$

In Table 2, we report the number of iterations and the CPU time (in parenthesis) for the convergence. As we see the preconditioner P_4 is the best.

Example 2. We consider the previous example with

$$-(u_{xx} + u_{yy}) + 2e^{x+y}(xu_x + yu_y) = f(x, y), \quad \text{in} \quad \Omega = (0, 1) \times (0, 1).$$

All of the assumptions are the same as the previous example. In Table 3 the spectral radii of the AOR iterative method and in Table 4 numerical results of

Table 1: Comparison of spectral radii for Example 1.

N	(γ, ω)	P_0	P_1	P_2	P_3	P_4
5	$(0.7, 0.8)$	0.8317	0.7964	0.7404	0.7486	0.6323
5	$(0.8, 1)$	0.7739	0.7305	0.6540	0.6444	0.5138
10	$(0.7, 0.8)$	0.9474	0.9350	0.9125	0.9116	0.8677
10	$(0.8, 1)$	0.9289	0.9135	0.8821	0.8815	0.8221

Table 2: Number of iterations and the CPU time for the convergence of the GMRES(10) for Example 1.

N	P_0	P_1	P_2	P_3	P_4
50	80(0.34)	57 (0.31)	33(0.22)	44(0.74)	29(0.17)
100	326(5.33)	130(2.83)	132(3.30)	110(2.75)	78 (1.97)
150	702(29.08)	365(19.73)	244(15.56)	350(23.02)	185(12.08)

the GMRES(10) method applied to the preconditioned systems $P_iAx = P_ib$, $i = 0, \ldots, 4$ are given. As we observe preconditioner P_4 is the best one among the chosen preconditioners.

4. Conclusion

For a class of matrices, we have shown that among the preconditioners of the form $\tilde{P} = I + L(\alpha) + U(\alpha)$ with $\alpha_{ij} \in [0, 1]$, the preconditioner $\hat{P} = I + L + U$ is the best one to speed up the convergence rate of the AOR iterative method. Numerical results of the AOR and GMRES(m) methods applied to different preconditioned systems confirm the presented theoretical results.

Table 3: Comparison of spectral radii for Example 2.

N	(γ, ω)	P_0	P_1	P_2	P_3	P_4
5	$(0.7, 0.8)$	0.8657	0.8358	0.7871	0.8049	0.6907
5	$(0.8, 1)$	0.8193	0.7823	0.7154	0.7033	0.5891
10	$(0.7, 0.8)$	0.9581	0.9481	0.9298	0.9306	0.8929
10	$(0.8, 1)$	0.9434	0.9309	0.9053	0.9045	0.8558

Table 4: Number of iterations and the CPU time for the convergence of the GMRES(10) for Example 2.

N	P_0	P_1	P_2	P_3	P_4
50	57(0.27)	46 (0.27)	28(0.19)	36 (0.22)	23(0.13)
60	85(0.52)	57 (0.45)	34(0.31)	49 (0.44)	29(0.25)
70	92(0.72)	79(0.89)	49(0.61)	54 (0.72)	37(0.47)
80	111(1.23)	84(1.23)	52(0.84)	94 (1.55)	45(0.73)

References

[1] O. Axelsson, Iterative solution methods, Cambridge University Press, Cambridge, 1996.

[2] A. Berman and R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979, SIAM, Philadelphia, PA, 1994.

[3] D.J. Evans, M.M. Martins and M. E. Trigo, The AOR iterative method for new preconditioned linear systems, J. Comput. Appl. Math. **132** (2001) 461-466.

[4] A. Hadjidimos, Accelerated overrelaxation method, Math. Comput. **32** (1978) 149-157.

[5] T.Z. Huang, G.H. Cheng, D.J. Evans and X.Y. Cheng, AOR type iterations for solving preconditioned inear systems, Int. J. Comput. Math. **82** (2005) 969-976.

[6] H. Kotakemori, K. Harada, M. Morimoto and H. Niki, A comparison theorem for the iterative method with the preconditioner $(I + S_{\max})$, J. Comput. Appl. Math. **145** (2002) 373-378.

[7] W. Li, A note on the preconditioned Gauss-Seidel (GS) method for linear systems, J. Comput. Appl. Math. **182** (2005) 81-90.

[8] W. Li, The convergence of the modified Gauss-Seidel methods for consistent linear systems, J. Comput. Appl. Math. **154** (2003) 97-105.

[9] W. Li and W. Sun, Modified Gauss-Seidel type methods and Jacobi type methods for Z-matrices, Linear Algebra Appl. **317** (2000) 227-240.

[10] M. Neumann and R.J. Plemmons, Convergence of parallel multisplitting iterative methods for M-matrices, Linear Algebra Appl. **88-89** (1987) 559-573.

[11] Y. Saad and M.H. Schultz, GMRES: A generalized minimal residual method for solving nonsymmetric linear systems, SIAM J. Sci. Statist. Comput. **7** (1986) 856-869.

[12] Y.Z. Song, Comparisons of nonnegative splittings of matrices, Linear Algebra Appl. **154-156** (1991) 433-455.

[13] L. Wang and Y.Z. Song, Preconditioned AOR iterative method for M-matrices, J. Comput. Appl. Math. **226** (2009) 114-124.

[14] M. Wu, L. Wang and Y. Song, Preconditioned AOR iterative method for linear systems Appl. Numer. math. **57** (2007) 672-685.

[15] Y. Zheng and T.Z. Huang, Modified iterative method for nonnegative matrices and M-matrices linear systems, Comput. Math. Appl. **50** (2005) 1587-1602.